Designing Tasks for Visualization and Reasoning in Dynamic Geometry Environment[1]

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Introduction

One of the important aims of the Hong Kong Secondary Mathematics Curriculum is to develop students' ability to conceptualize, inquire, reason and communicate (Curriculum Development Council, 1999, p.4). Hence the terms "explore" and "justify" appear in the learning objectives of many topics. For example, it is expected that students could "explore the formula for the area of circles" (ibid., p.20), or could "explore and justify the methods of constructing centres of a triangle such as in-centre, circumcentre, orthocentre, centroids, etc." (ibid., p.23).

However, it seems that the learning tasks proposed in Hong Kong textbooks could not always fulfill this aim. For instance, when constructing the circumcircle of a triangle, students are told directly to first construct the perpendicular bisectors of the three sides using rulers and compasses, and then use their intersection as a centre to draw a circle passing through one of the vertices, and finally see that this circle also passes through the other two vertices (Figure 1). In this task students are neither provided the opportunity to explore nor asked to justify how the circumcircle could be constructed, but instead just to verify the correctness of the procedures to construct the circumcircle given by the textbooks.

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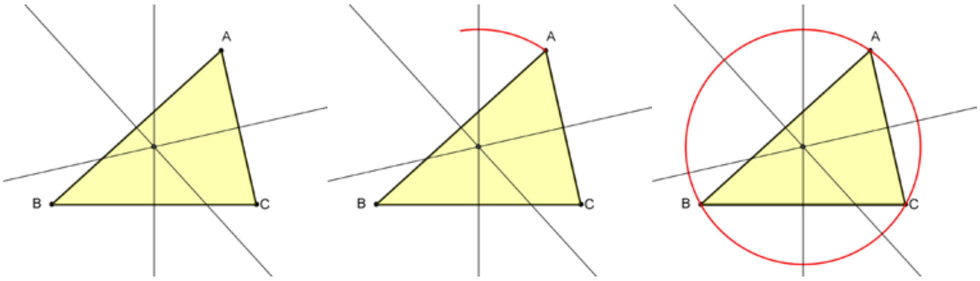


Figure 1

In this article, I am going to present some explorative tasks designed in GeoGebra, a powerful dynamic geometry (DG) freeware, that could facilitate justification through visualization. I shall propose a framework on task design in dynamic geometry environment (DGE) to facilitate visualization and reasoning based on Duval's model of the role of visualization in the development of geometrical reasoning (Duval, 1998). In particular, I suggest that the use of soft constructions (Healy, 2000) is an effective approach for designing tasks to foster operative apprehension for visualization and reasoning in DGE.

Duval's Model of Geometrical Reasoning

Duval (1998) suggests that geometry involves three kinds of closely connected cognitive processes fulfilling specific epistemological functions, namely, *visualization*, *construction* and *reasoning*. Their epistemological functions and connections are represented by Figure 2 below.

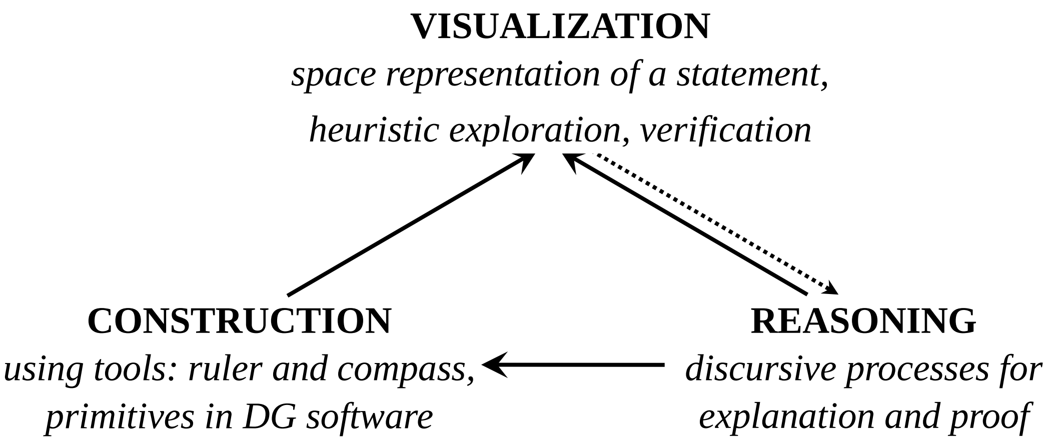


Figure 2

In the figure each arrow represents the way a kind of cognitive process can support another kind in a task. The dotted arrow suggests that visualization does not always help reasoning. For example, visualization can be misleading if our visualized image is a special case. Duval states that these three kinds of cognitive processes are quite different and must be developed separately, and the significance of the teaching of geometry is to develop visual representation and reasoning abilities and to favour the synergy of these processes.

To facilitate visualization as well as reasoning, Duval suggests the necessity of a kind of apprehension of geometric figures called *operative apprehension*, which means operations on the figure or its subfigure, either mentally or physically, that gives insight into the solution of a problem. He emphasizes that operative apprehension is crucial and teachers have to identify factors triggering or inhibiting it so as to make visualization possible and gives rise to various transfers.

With regard to the use of dynamic geometry software (DGS), Duval states that DGS provides enormous possibilities of visualization through the introduction of the aspect of movement, and allows manipulations of geometric objects and hence true explorations of geometrical situations. However, the construction-centered design of DGS does not develop all functions of visualization, in particular the operative apprehension.

Duval's theory emphasizes the importance of operative apprehension to facilitate visualization and also reasoning in the teaching and learning of geometry. In view of his comments on the uses and limitation of DGS in visualization, I shall discuss how to design tasks in DGE to foster operative apprehension for visualization and reasoning. I would first define what operative apprehension means in the DGE, and how the use of soft constructions proposed by Healy (2000) could be an effective approach to designing task to foster operative apprehension.

Operative Apprehension in DGE

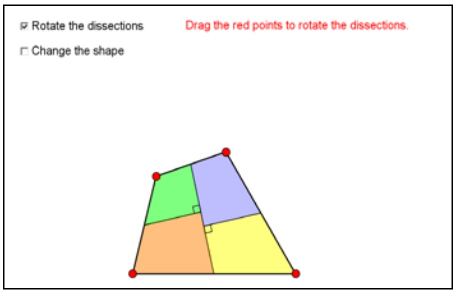
A task is a set of pre-designed, environmentally situated materials aiming to engage learners in activities that could transform the ways they see and do mathematics (Leung, 2011). A task has to be pre-designed in the way that through these pre-designed means, learners are guided to construct insights and the meaning of the mathematics knowledge. A task is also environmentally situated, in the sense that the qualities or tools of the environment have been made use of to empower learners with extended or amplified abilities to acquire knowledge which could not be acquired in the same ways as in other environments (Leung, 2011). In what follows, I shall discuss how to design tasks situated in DGE to foster operative apprehension for visualization and reasoning in Duval's framework. In particular, I will focus on pre-designed DG figures and interpret Duval's operative apprehension as the following:

> Operative apprehension of a mathematical concept or problem in DGE is the insights into the concept or the solution of the problem revealed by operating on a pre-designed figure in the environment through dragging.

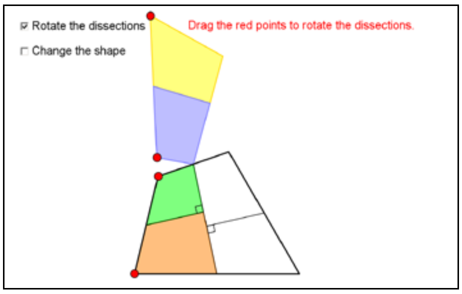
Let me illustrate the significance of operative apprehension in DGE using a task I designed in GeoGebra. This task originates from the following problem in a textbook.

> *A quadrilateral is dissected by a line joining the mid-points of one pair of opposite sides, and the perpendiculars to this line from the mid-points of the other pair of opposite sides.* (See Figure 3(a).) *What shape can you get from this dissection?*

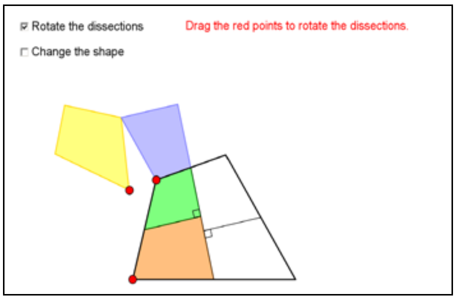
A task is designed in GeoGebra to help learners to explore this problem (http://www.geogebratube.org/student/m3459). In this task, a quadrilateral is dissected into four pieces as described in the problem. Each piece can be rotated through dragging the red point at the vertex. In this way we can see that how the four pieces could form a rectangle (Figure 3). Also, the operation gives us the insights to reason why this dissection gives a rectangle, by, for instance, thinking about why the four angles at the vertices give a sum 360° (Figure 3(d)).



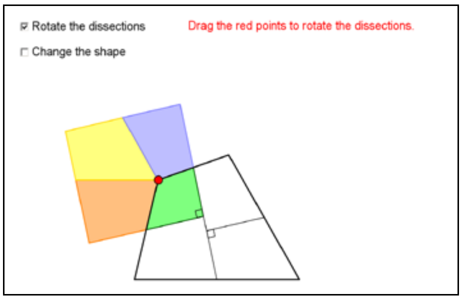
(a)



(b)



(c)



(d)

Figure 3

Besides rotating the four pieces, learners can also operate on the shape of the quadrilateral. After checking the "Change the shape" box, four green points appear at the vertices of the quadrilateral and the shape of the quadrilateral could be changed by dragging them. Through dragging the vertices, I see that the dissection would give a square for some shapes of the quadrilateral (Figure 4).

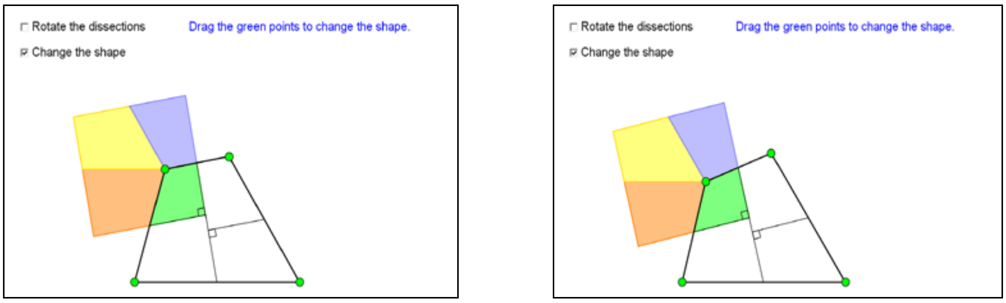


Figure 4

This problem reminds me the famous Haberdasher Puzzle composed by English mathematician Dudeney (Dudeney, 1907). This puzzle shows how an equilateral triangle could be dissected into a square (Figure 5). Although I have known this puzzle for a long time, I never understand how Dudeney could think of this method of dissection, nor have any idea how to generalize his method to dissect an arbitrary triangle into a square.

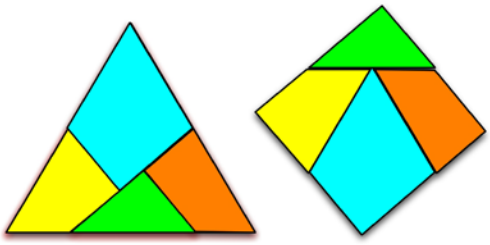
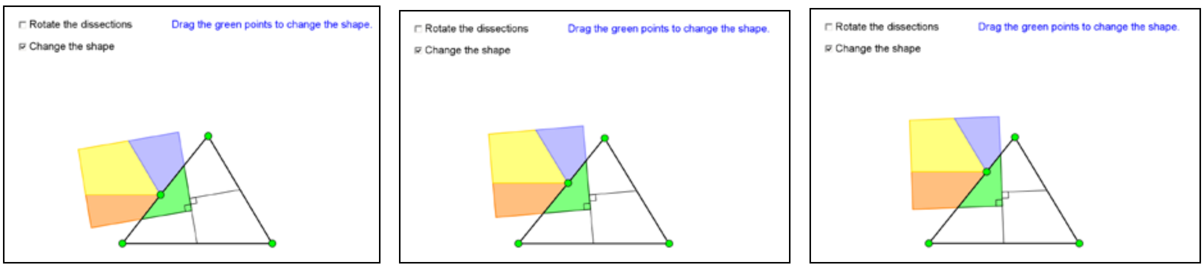


Figure 5

When I try to compare the quadrilateral problem with Dudeney's puzzle, I suddenly realize that if I drag a vertex, say the upper-left one, to a position at which it is collinear with the other two adjacent vertices (Figure 6(a)), the quadrilateral would be degenerated into a triangle which is dissected into a rectangle. Furthermore, if I drag this vertex along the side of the triangle (Figure 6(b)), the shape of this triangle is unchanged and the area of the rectangle is kept constant while its length and width are decreasing and increasing respectively through dragging. Hence I should get a square somewhere on this side (Figure 6(c)).



(a) (b) (c)

Figure 6

After the above exploration I see how an arbitrary triangle could be dissected into rectangles of various sizes, and there should be a particular dissection that gives a square. Through operating on the shape of the quadrilateral, I get important insights of comprehending how a general triangle could be dissected into a rectangle, and investigating when the dissection would give a square.

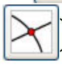
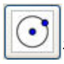
This example illustrates the advantage of fostering operative apprehension in DGE. If we use a paper quadrilateral, although we could cut it to see how it could be dissected into a rectangle, it is impossible for us to operate on its shape. In DGE we can operate on the shape of the quadrilateral so that we can degenerate it to a triangle to get the insights of how a triangle could be dissected into a rectangle.

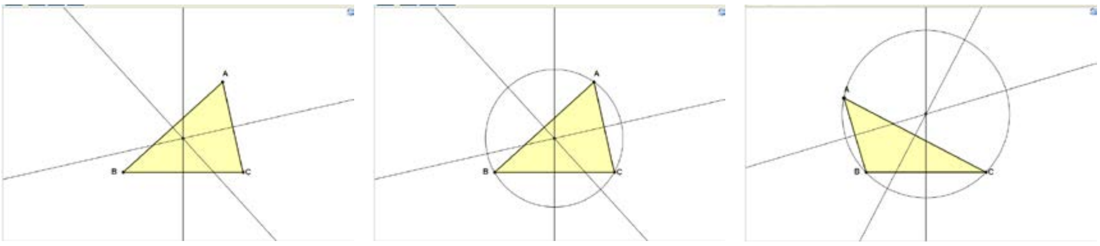
Operative Apprehension for Visualization and Reasoning: Soft Construction

At the beginning of research in dynamic geometry, tasks in robust constructions, i.e. constructions preserve relationships upon dragging, were recognized as promoting for the learning of geometry. However, Healy (2000) discovered through observation that, rather than robust constructions, students preferred to investigate constructions "in which one of the chosen properties is purposely constructed by eye, allowing the locus of permissible figures to be built up in an empirical manner under the control of the student". Healy called these constructions *soft constructions*.

Healy differentiates the roles of dragging in robust and soft constructions. In a robust construction, dragging provides a visual verification of the validity of the construction through dragging. In a soft construction, dragging is not verification but part of the construction itself. Through dragging, the general can emerge from the specific by searching empirically for the locus of figures fulfilling the given conditions. Soft constructions offer a transition from an empirical approach to a theoretical approach in solving a geometry problem.

In the lens of Duval's model of geometrical reasoning, tasks in robust and soft constructions can be considered as operative apprehension on figures serving different functions of visualization: a robust construction provides a verification of the construction, while a soft construction provides heuristics or insights through an empirically searched locus which mediates reasoning. I shall illustrate this point with two GeoGebra tasks of drawing the circumcircle of a triangle, one in robust construction and one in soft construction.

In the robust construction task, perpendicular bisectors of the three sides are first constructed using the "Perpendicular Bisector" tool . A circle centred at their intersection (found by the "Intersection" tool ) and passing through either one vertex (say *A*) is constructed using the "Circle" tool , and it could be seen that this circle also passes through the other two vertices (*B* and *C*). By dragging the vertices of the triangle, learners can check the validity of the construction by seeing that the circle always passes through the vertices. They can also see that the circumcentre lies outside the triangle when the triangle is obtuse (Figure 7).



(a) (b) (c)

Figure 7

In the soft construction task (http://www.geogebratube.org/student/m3958), learners are first given the triangle and a circle which can be moved by dragging its centre (in red) and a blue point on its circumference (Figure 8(a)). Learners first drag the blue point to either one vertex, say *A*, and a dotted line joining *A* and the centre would then be shown (Figure 8(b)). Then they drag the red centre to different positions at which the circle also passes through another vertex *B*, and when this happens a dotted line joining *B* and the centre would be shown. These positions of the centre are marked in red, and learners can see that the locus of the centres of circles passing through *A* and *B* is a straight line (Figure 8(c)). Learners can then be asked what this line of locus should be, and the two dotted lines from the centre to *A* and *B* providing hints for them to reason that this line is the perpendicular bisector of *AB* (through looking at two congruent triangles). Once they recognize that the locus of the centres should be the perpendicular bisector, they can find empirically the loci of the centres when the circle passes through *A*, *C* (Figure 8(d)) and *B*, *C* (Figure 8(e)), and finally visualize that the circumcircle should centre at the intersection of the three loci, i.e. the intersection of the perpendicular bisectors of the three sides (Figure 8(f)).

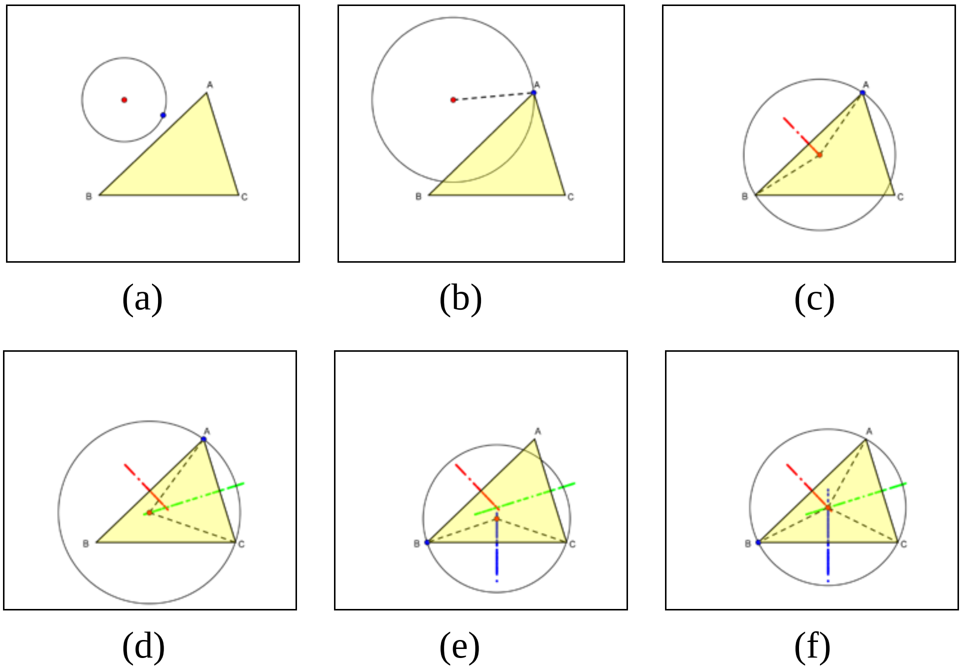


Figure 8

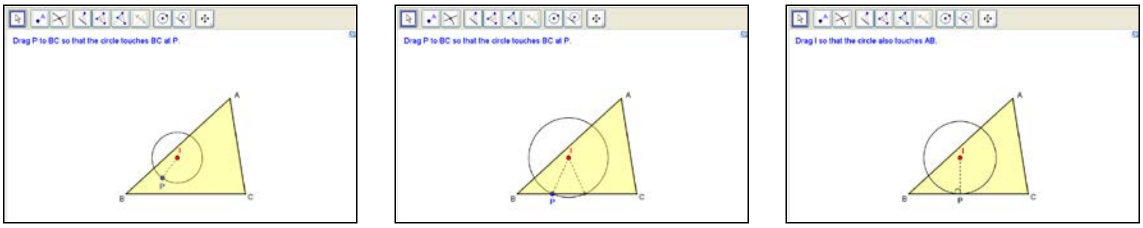
The above example illustrates how a task in soft construction could foster operative apprehension by recording the loci of positions at which the eye construction satisfies the given conditions. These loci of positions provide insights to solve the problem, and also mediate the reasoning of why the problem could be solved in this way. I now propose the following principle of using soft constructions to design task fostering operative apprehension for visualization and reasoning in DGE.

Principle of using soft constructions to foster operative apprehension

> Learners are provided opportunities to perform soft (eye) construction by dragging. The loci of the dragging satisfying the given conditions, together with the other elements supporting their visualization and reasoning, would be shown to the learners so that theoretical elements could emerge from the empirical evidences.

I further elaborate the above principle using a more sophisticated task of finding the incircle of a triangle (http://www.geogebratube.org/student/m4363).

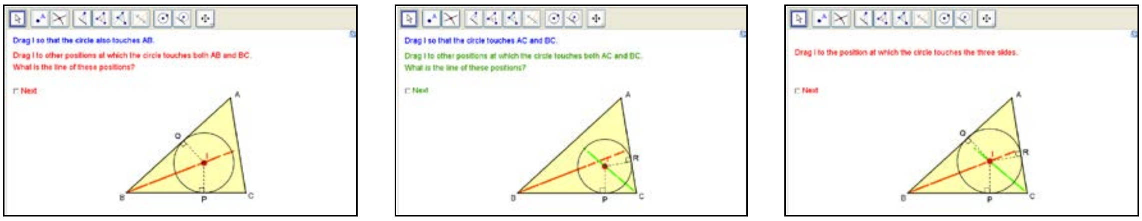
In this task the triangle and a circle of centre *I* and passing through *P* are given, and the radius *IP* is also shown. Learners are first asked to drag *P* to the side *BC*, then another dotted line would be shown to indicate that there are two intersections (Figure 9(b)). By dragging *P* towards the other intersection learners would visualize that for the circle to touch *BC*, the two dotted radii should overlap to form one radius *IP* perpendicular to *BC* (Figure 9(c)). I also anticipate that this process of dragging, together with the overlapping of the two radii, would help learners to reason why the tangent of a circle should be perpendicular to the radius.



(a) (b) (c)

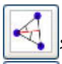
Figure 9

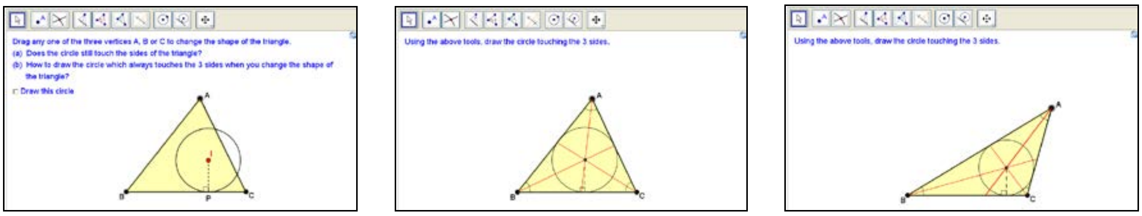
Once the circle touches *BC*, *P* can no longer be dragged and learners are asked to drag the centre *I* to different positions so that the circle would also touch *AB*, and the locus of *I* is marked in red (Figure 10(a)). Learners are prompted to identify this line of locus as the angle bisector at *B*, and could explain this by looking at the congruent triangles *IBP* and *IBQ*. Similarly learners identify the locus of *I* at which the circle touches *BC* and *AC* as another angle bisector at *C* (Figure 10(b)), and see that the circle would touch the three sides when *I* is at the intersection of the angle bisectors (Figure 10(c)).



(a) (b) (c)

Figure 10

Finally, the three vertices of the triangle are made draggable to the learners and they are asked to drag the vertex *A* to change the shape of the triangle, and see that the original circle no longer touches the three sides (Figure 11(a)). They are then asked to perform robust construct of the incircle by constructing the suitable lines in a triangle using the given tools (median , angle bisector , altitude  and perpendicular bisector ) and the touching circle tool  (Figure 11(b)). They can then check the validity of their construction by dragging the vertices (Figure 11(c)).



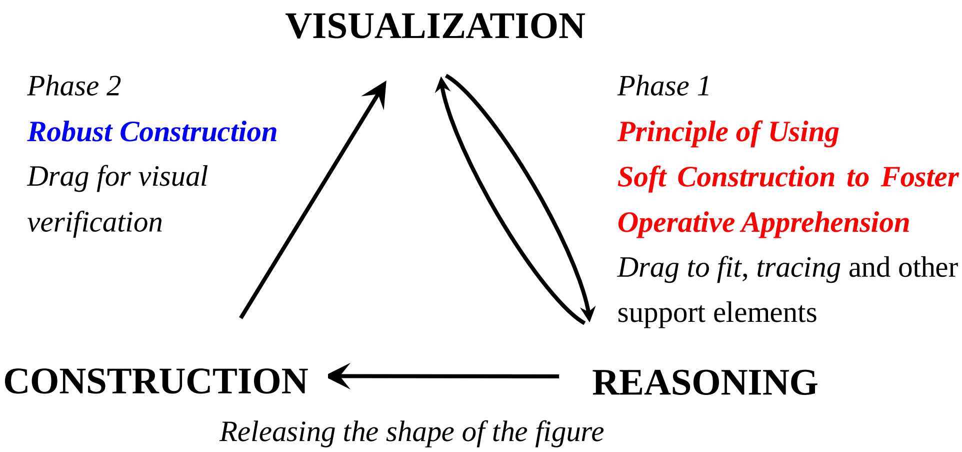
(a) (b) (c)

Figure 11

Discussions and Implications

Based on the above illustrations, I propose a model of task design in DGE to foster operative apprehension for visualization and reasoning by modifying Duval's model of geometrical reasoning as follows:

Task Design Model in DGE for
Visualization and Reasoning through Dragging



Task design in this model consists of two phases. In Phase 1, the *Principle of Using Soft Construction to Foster Operative Apprehension* is applied so as to foster students' operative apprehension through *soft construction*, i.e. to use the *drag to fit* strategy to find solutions satisfying the given conditions. In the process of soft construction, the *trace* of the *locus of validity* (Leung and Lopez-Real, 2002) and other support elements that mediate reasoning would be shown. Use my in-centre task as an example (p.7), the dotted radii and their overlapping through dragging (Figure 9) are the support elements which are shown to students to mediate the insight and reasoning of perpendicularity of the radius and the side when the circle touches it. Similarly, the traces and the radii shown by the software in Figure 10 support the reasoning that the traces are the angle bisectors of the triangle and that in-centre lies on their intersection.

In this phase dragging and tracing are the *cognitive tools* (Leung, 2011) to start a recursive cycle between visualization and reasoning until a solution and its justification is reached. In the design of the in-circle task, students are guided to first visualize through dragging that the radius has to be perpendicular to the side when the circle touches it. Then with this property students are further guided to visualize through dragging and tracing that centre of the circle must lie on a certain line when the circle touches two sides of the triangle. They are then guided to reason, using the trace and the dotted radii, and explain that this line is in fact the angle bisector. Finally they further visualize that when the centre lies on the intersection of the two angle bisectors, the circle would touch all the three sides and at this stage they should be able to explain why this happens.

When the solution and its explanation are reached in Phase 1, the task is then transited to Phase 2 in which students are required to use the construction tools given by the software to do a *robust* construction to verify the solution and explanations they obtained in Phase 1. This is done by releasing the shape of the figure in the problem so that students observe that the soft construction in Phase 1 no longer works when the shape of the figure is changed (Figure 11(a)). Students are then asked to use the tools of the software to construct a robust in-circle that always touch the three sides (Figure 11(b)(c)). In this phase dragging is a tool for visual verification of the construction.

This model shows how the different roles of robust and soft constructions could foster operative apprehension, through which the synergy of visualization, reasoning and construction can be facilitated. If we agree with Duval that developing visualization and reasoning abilities to favour the synergy of the three cognitive processes is of crucial importance for the teaching of geometry, designing tasks to foster operative apprehension for visualization and reasoning in DGE effectively would then be very promising to promote the teaching of geometry. This is also a great challenge to all teachers, educators and researchers. It is hoped that the principle and the model of task design in DGE proposed in this article could provide a useful initiation for further discussions, challenges and refinement in future task design research.

References

Curriculum Development Council (1999). *Syllabuses for Secondary Schools: Mathematics (Secondary 1 – 5)*. Hong Kong: The Education Department.

Dudeney, H. (1907). *The Canterbury Puzzles*. London: Nelson, 1907. Reprinted Mineola, NY: Dover, 1958.

Duval, R. (1998). Geometry from a Cognitive Point of View. In C. Mammana & V. Villani (Eds.), *Perspectives on the Teaching of Geometry fro the 21st Century* (p.37–52). Dordrecht: Kluwer Academic Publishers.

Healy, L. (2000). Identifying and explaining geometrical relationship: interactions with robust and soft Cabri constructions. In T. Nakahara and M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education*, (Vol.1, pp. 103–117). Hiroshima: Hiroshima University.

Leung, A. & Lopez-Real, F. (2002). Theorem justification and acquisition in dynamic geometry: A case of proof by contradiction. *International Journal of Computers for Mathematical Learning*, 7, pp. 145-165.

Leung, A. (2011). An epistemic model of task design in dynamic geometry environment. *ZDM Mathematics Education*, Vol.43, pp. 325–336.

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