Pedagogical Design Focusing on Learning Difficulties – Area of Circles

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What is the Area of a Circle Exactly?

One of the important aims of the Hong Kong Secondary Mathematics Curriculum is to develop students' ability to conceptualize, inquire, reason and communicate (Curriculum Development Council, 1999 p.4). In the learning and teaching of the idea of areas and volumes, one of the learning objectives is to "explore the formula for the area of a circle". The usual task offered by textbooks is to divide a circle of radius r into a certain number of equal parts (say 16, see Figure 1), and then rearrange them to form a figure close to a parallelogram (Figure 2). Students are then asked to find the height (r) and base (πr) of this "parallelogram", from which the area is found to be $\pi r \times r = \pi r^2$. Hence the area of the circle is also πr^2.

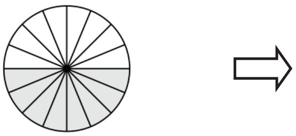


Figure 1

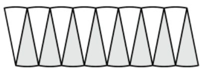


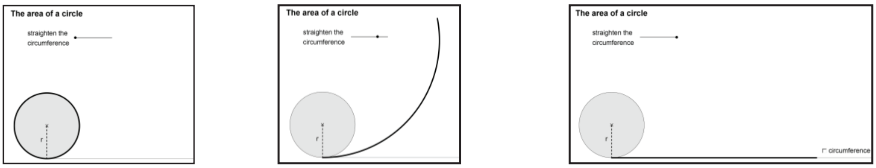
Figure 2

The above task may give the impression to students that πr^2 is just an approximation instead of an exact value of the area. As we all know, πr^2 is the limit of the area of the rearranged figure when we increase the number of divisions indefinitely, and in this sense πr^2 is not just an approximation but an exact value. The limitation of the physical manipulative (the paper circle) makes it difficult for students to perceive this idea, and the use of virtual manipulative may help to fill in this epistemic gap. In this article, I am going to present a task I design in GeoGebra to help students to explore the area of a circle by dissecting it into a parallelogram-like figure, and then visualize the limit of the shape dynamically by increasing the number of divisions.

Introduction to the Task

The task consists of a Java applet named "circle-area.html" which is designed using the free dynamic geometry software GeoGebra. The applet can be run in any browser with Java plug-in without having GeoGebra installed. This applet and its complementary worksheets can be downloaded from the Education Bureau's Web-based Learning and Teaching support website (http://wlts.edb.hkedcity.net) by clicking the corresponding links in "What's Hot". The stand-alone applets can also be accessed and downloaded in GeoGebraTube.
(English version: http://www.geogebratube.org/material/show/id/279
Chinese version: http://www.geogebratube.org/material/show/id/550)

In the applet, students are given a circle of radius *r* (Figure 3(a)). Since they have learnt the formula for the circumference which is crucial for this task, as a revision they are first asked to straighten the circumference using the slider (Figure 3(b)) and then to give the circumference ($2\pi r$) (Figure 3(c)).



(a) (b) (c)

Figure 3

Students then check the "dissect" box to dissect the circle into 12 equal parts (Figure 4(a)), and drag the green point on the "rearrange" slider to rearrange these parts to form a parallelogram-like figure, as shown in Figure 4(b) to Figure 4(e).

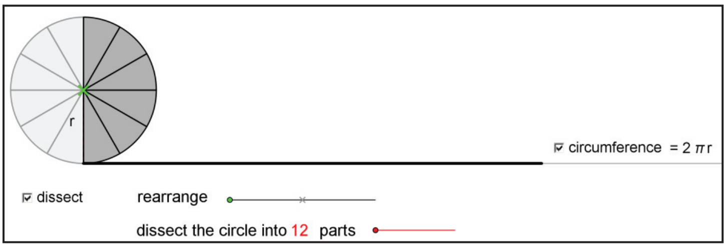


Figure 4 (a)

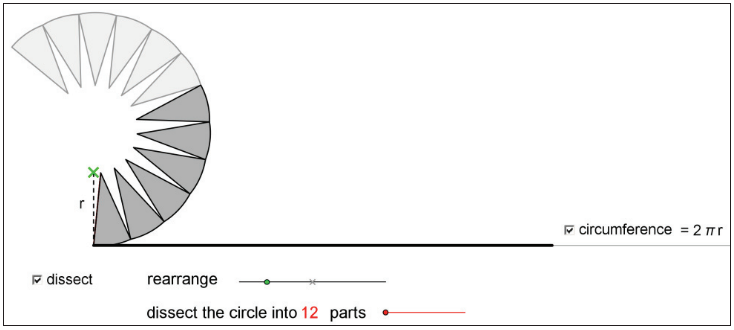


Figure 4 (b)

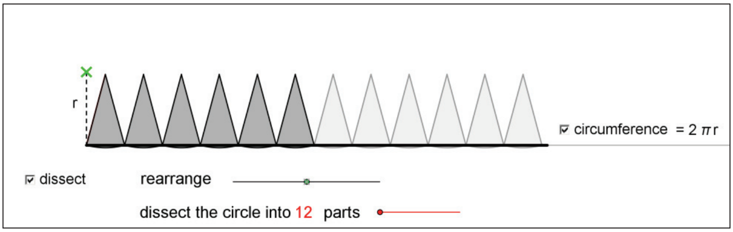


Figure 4 (c)

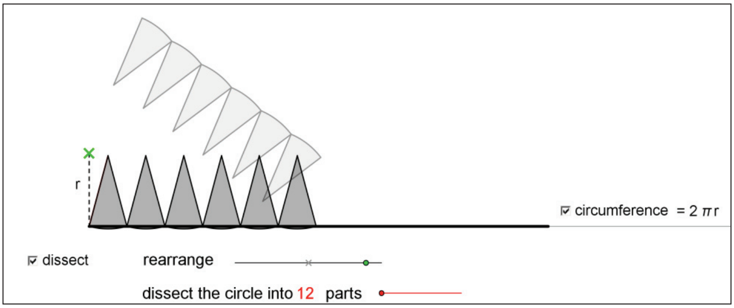


Figure 4 (d)

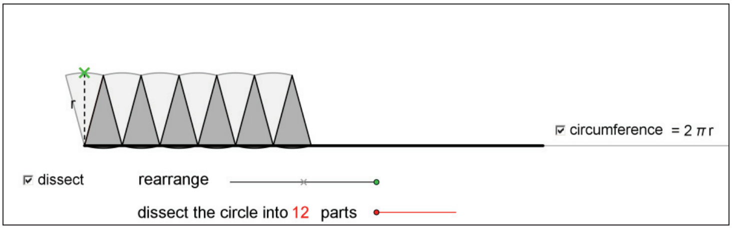


Figure 4 (e)

Here students are asked which kind of shapes they think Figure 4(e) looks like. They can then drag the red point on the red slider to increase the number of divisions gradually to 50 (Figure 5(a)), 100 (Figure 5(b)) and 200 (Figure 5(c)) so that students can see the changes of the shape of the rearranged figure when the number of divisions increases.

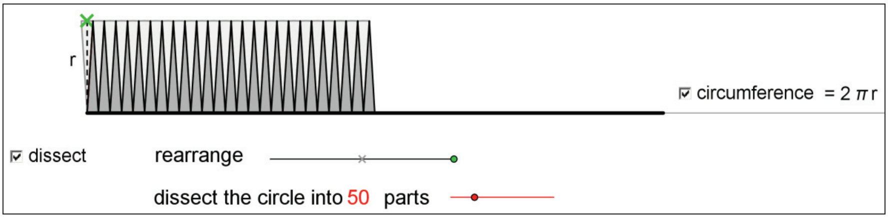


Figure 5 (a)

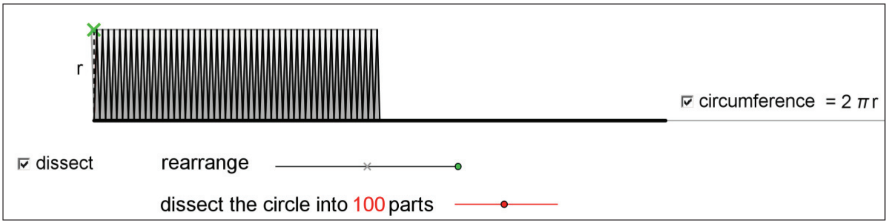


Figure 5 (b)

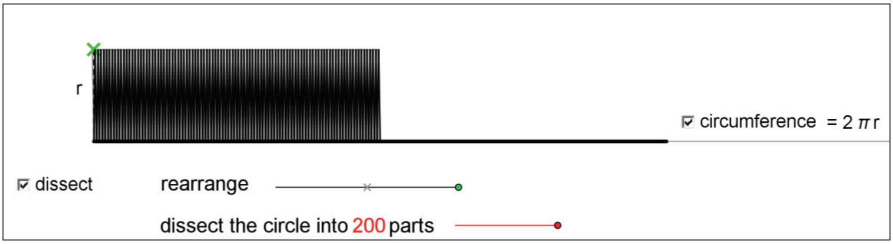


Figure 5 (c)

Now students are asked to think about the crucial question:

"*As the number of parts becomes larger and larger, what kind of shape is the figure closer and closer to?*"

Through the dynamic applet, students are expected to see that as the number of parts increases, the shape of the figure becomes closer to a rectangle. Students are then required to find the area of this rectangle.

The width of this rectangle is equal to the radius of the circle r. To find the length of the rectangle, students can reverse the figure back to the circle (Figure 6(a)), and then rearrange the parts to the positions as shown in Figure 6(b). They can then see that the length of the rectangle should be half of $2\pi r$, the circumference of the circle. Hence the area of the rectangle, and also that of the circle, is $\pi r \times r = \pi r^2$.

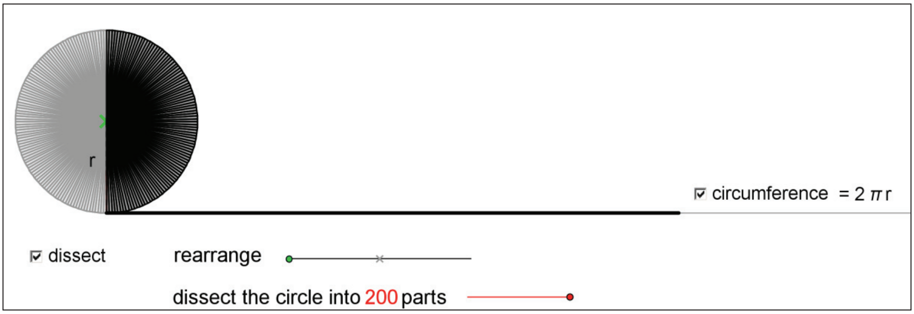


Figure 6 (a)

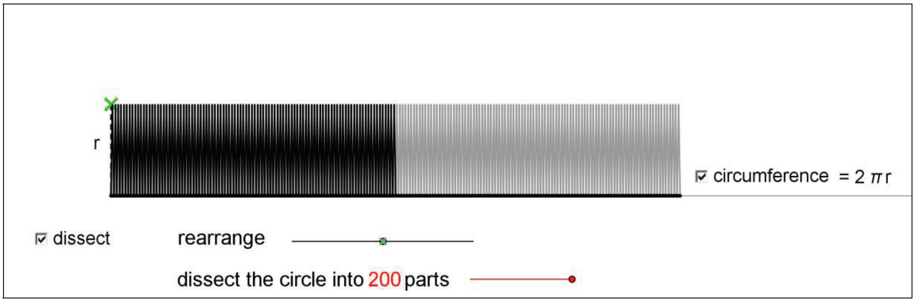


Figure 6 (b)

Mathematics in the Design

In this section I am going to share the mathematics involved in the design of this task and the corresponding commands in GeoGebra. Readers who are not interested in these technical details could skip this section.

When I design the task, I first think about how to lay down the sectors of the circle evenly as shown in Figure 4(b). Consider the vertices of an inscribed regular $2n$-sided polygon, and let the angle at the centre of each sector be $\theta\ (= \frac{2\pi}{2n} = \frac{\pi}{n})$. Note that each of the exterior angles of the polygon is also θ, as indicated in Figure 7(a). I construct the sequence of the vertices of this polygon (named list1) in GeoGebra using the following command:

list1 = Sequence[(sin(j θ), 1 - cos(j θ)), j, 0, 2 n]

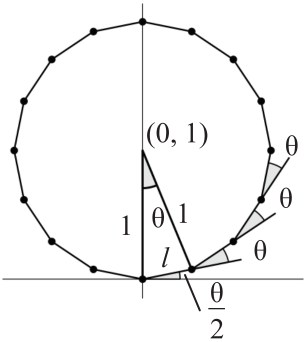


Figure 7 (a)

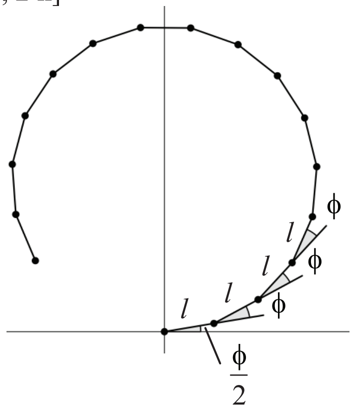


Figure 7 (b)

I imagine that when the exterior angles in Figure 7(a) are decreased gradually to zero, the perimeter of the polygon can be laid down to a straight line. I therefore introduce a variable k whose value (from 0 to 1) is controlled by the first half of the slider "rearrange". I also introduce another variable $\phi = (1 - k)\theta$, which decreases from θ to 0 when k is increased from 0 to 1, and the perimeter of the polygon would then be laid down to a straight line as shown in Figure 7(b). The j th vertex (x_j, y_j) $(j = 0, 1, 2, \cdots, 2n)$ of the polygon in Figure 7(b) is given by the following:

$$\begin{aligned} x_j &= l\,(\cos\frac{\phi}{2} + \cos(\phi + \frac{\phi}{2}) + \cos(2\phi + \frac{\phi}{2}) + \ldots + \cos((j-1)\phi + \frac{\phi}{2}) \\ &= l\,(\cos\frac{\phi}{2} + \cos\frac{3\phi}{2} + \cos\frac{5\phi}{2} + \ldots + \cos\frac{(2j-1)\phi}{2}); \\ y_j &= l\,(\sin\frac{\phi}{2} + \sin(\phi + \frac{\phi}{2}) + \sin(2\phi + \frac{\phi}{2}) + \ldots + \sin((j-1)\phi + \frac{\phi}{2}) \\ &= l\,(\sin\frac{\phi}{2} + \sin\frac{3\phi}{2} + \sin\frac{5\phi}{2} + \ldots + \sin\frac{(2j-1)\phi}{2}). \end{aligned}$$

In the above, $l = 2\sin\frac{\theta}{2}$ is the length of the side of the polygon. We see that the coordinates of each vertex are the sums of trigonometric series. These sums can be evaluated using complex numbers as follows.

$$x_j + i\,y_j = l\left(\operatorname{cis}\frac{\phi}{2} + \operatorname{cis}\frac{3\phi}{2} + \operatorname{cis}\frac{5\phi}{2} + \ldots + \operatorname{cis}\frac{(2j-1)\phi}{2}\right)$$

$$= l\left(e^{i\frac{\phi}{2}} + e^{i\frac{3\phi}{2}} + e^{i\frac{5\phi}{2}} + \ldots + e^{i\frac{(2j-1)\phi}{2}}\right)$$

$$= l\frac{e^{i\frac{\phi}{2}}\left(e^{j\cdot i\phi}-1\right)}{e^{i\phi}-1}$$

$$= l\frac{\left(\cos\frac{\phi}{2}+i\sin\frac{\phi}{2}\right)\left(\cos j\phi + i\sin j\phi - 1\right)}{\cos\phi + i\sin\phi - 1}$$

$$= l\frac{\left(\cos\frac{\phi}{2}+i\sin\frac{\phi}{2}\right)\left(-2\sin^2\frac{j\phi}{2}+2i\sin\frac{j\phi}{2}\cos\frac{j\phi}{2}\right)}{-2\sin^2\frac{\phi}{2}+2i\sin\frac{\phi}{2}\cos\frac{\phi}{2}}$$

$$= l\frac{\cancel{\left(\cos\frac{\phi}{2}+i\sin\frac{\phi}{2}\right)}\cdot \cancel{2i}\sin\frac{j\phi}{2}\left(\cos\frac{j\phi}{2}+i\sin\frac{j\phi}{2}\right)}{\cancel{2i}\sin\frac{\phi}{2}\cancel{\left(\cos\frac{\phi}{2}+i\sin\frac{\phi}{2}\right)}}$$

$$= l\left(\frac{\sin\frac{j\phi}{2}\cos\frac{j\phi}{2}}{\sin\frac{\phi}{2}} + i\frac{\sin\frac{j\phi}{2}\sin\frac{j\phi}{2}}{\sin\frac{\phi}{2}}\right)$$

$$\therefore\quad x_j = l\,\frac{\sin\frac{j\phi}{2}\cos\frac{j\phi}{2}}{\sin\frac{\phi}{2}}\ ;\quad y_j = l\,\frac{\sin\frac{j\phi}{2}\sin\frac{j\phi}{2}}{\sin\frac{\phi}{2}}$$

Based on the above calculations, I use the following command to construct the sequence of the vertices in Figure 7(b) (named list2):

list2 = Sequence[If[k < 1, (*l* sin(j ϕ/2) cos(j ϕ/2) / sin(ϕ/2), *l* sin(j ϕ/2) sin(j ϕ/2) / sin(ϕ/2)), (*l* j, 0)], j, 0, 2 n]

When $k = 1$, $\phi = 0$ and the vertex $\left(l\,\frac{\sin\frac{j\phi}{2}\cos\frac{j\phi}{2}}{\sin\frac{\phi}{2}},\ l\,\frac{\sin\frac{j\phi}{2}\sin\frac{j\phi}{2}}{\sin\frac{\phi}{2}}\right)$ would become undefined. Therefore an IF command is added inside the SEQUENCE command to ask GeoGebra to construct the vertices ($j\,l$, 0) ($0 \le j \le 2n$, all vertices lie on the x-axis in this case) when $k = 1$.

With the sequence of vertices, we can now construct the sequence of sectors. The idea is to first translate each sector of the circle to the corresponding vertex on the laid-down perimeter, and then rotate it according to the angle as indicated in Figure 8. I use the following command to construct the sequence of the sectors list3 of the first semicircle (the part in deep grey in Figure 9):

list3 = Sequence[Rotate[Translate[CircularSector[(0, 1), Element[list1, j], Element[list1, j+1]], vector[Element[list1, j], Element[list2, j]]], -(2j-1) k θ/2, Element[list2, j]], j, 1, n]

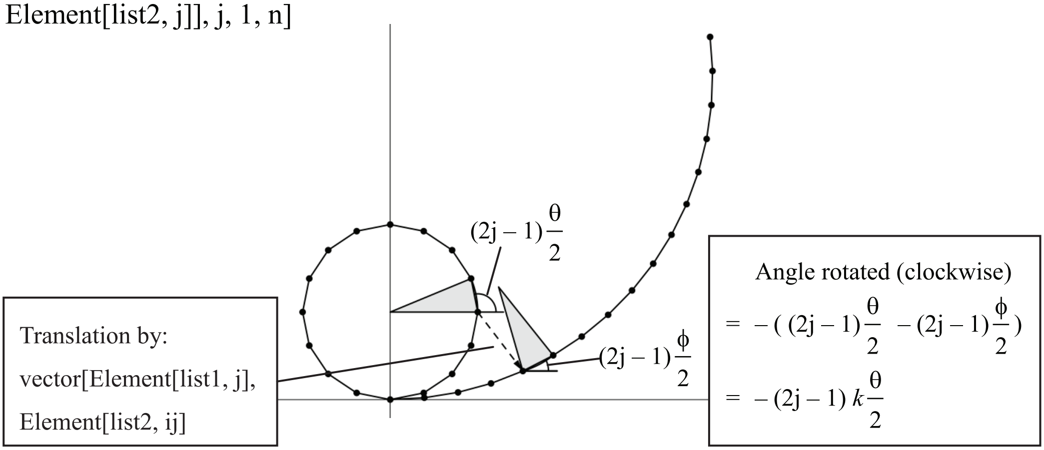


Figure 8

For the sectors of the other semicircle (the part in light grey in Figure 9), since they have to be rotated by 180° about the centre P, I introduce another variable k' whose value is controlled by the second half of the slider “rearrange”. The value of k' will be zero when k varies from 0 to 1. After k reaches the value 1, the value of k' will be increased form 0 to 1. The sequence of these sectors (list4) is constructed by the command

list4 = Sequence[Rotate[Rotate[Translate[CircularSector[(0, 1), Element[list1, j], Element[list1, j+1]], vector[Element[list1, j], Element[list2, j]]], -(2j-1) k θ/2, Element[list2, j]], 180° k', (n l - sin(θ/2)/2, cos(θ/2)/2)], j, n+1, 2n]

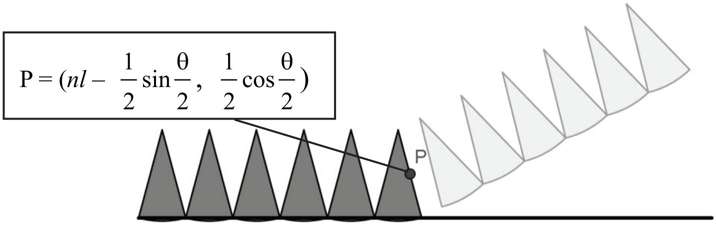


Figure 9

After solving the problem of laying down (and rotating) the sectors, I think about whether it is possible to open (straighten) the circumference similarly (Figure 3(a) to (c)). Consider the end point L when the circumference is opened to a certain position. This end point L should be the limit of the last vertex ($j = 2n$) in Figure 7(b) when n tends to infinity. Therefore,

$$L = \left(\lim_{n\to\infty} l\frac{\sin\frac{2n\phi}{2}\cos\frac{2n\phi}{2}}{\sin\frac{\phi}{2}},\ \lim_{n\to\infty} l\frac{\sin\frac{2n\phi}{2}\sin\frac{2n\phi}{2}}{\sin\frac{\phi}{2}}\right)$$

$$= \left(\lim_{n\to\infty} 2\sin\frac{\pi}{2n}\,\frac{\sin(1-k)\pi\cdot\cos(1-k)\pi}{\sin\frac{(1-k)\pi}{2n}},\ \lim_{n\to\infty} 2\sin\frac{\pi}{2n}\,\frac{\sin(1-k)\pi\cdot\cos(1-k)\pi}{\sin\frac{(1-k)\pi}{2n}}\right)$$

$$= \left(2\sin(1-k)\pi\cdot\cos(1-k)\pi\cdot\lim_{n\to\infty}\frac{\sin\frac{\pi}{2n}}{\sin\frac{(1-k)\pi}{2n}},\ 2\sin(1-k)\pi\cdot\cos(1-k)\pi\cdot\lim_{n\to\infty}\frac{\sin\frac{\pi}{2n}}{\sin\frac{(1-k)\pi}{2n}}\right)$$

By the L'Hospital's Rule, we have $\lim_{n\to\infty}\frac{\sin\frac{\pi}{2n}}{\sin\frac{(1-k)\pi}{2n}} = \frac{\lim_{n\to\infty}\frac{\pi}{2}\cos\frac{\pi}{2n}}{\lim_{n\to\infty}\frac{(1-k)\pi}{2}\cos\frac{(1-k)\pi}{2n}} = \frac{1}{1-k}$,

$$\therefore\quad L = \left(\frac{2\sin(1-k)\pi\cdot\cos(1-k)\pi}{1-k},\ \frac{2\sin(1-k)\pi\cdot\sin(1-k)\pi}{1-k}\right)$$

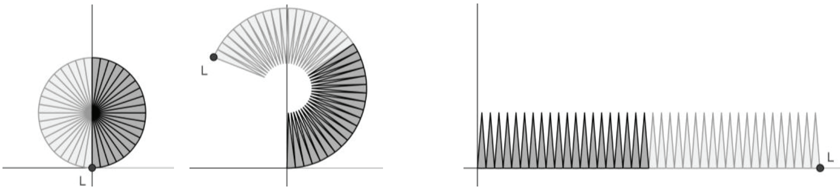
$$= \left(\frac{\sin 2(1-k)\pi}{1-k},\ \frac{1-\cos 2(1-k)\pi}{1-k}\right)$$

Since the circumference and the sectors are to be laid down separately, I introduce another variable m ($0 \le m \le 1$, controlled by the slider "straighten the circumference") to control the position of L. Hence the coordinates of L

are $(\frac{\sin 2(1-m)\pi}{1-m}, \frac{1-\cos 2(1-m)\pi}{1-m})$, and the command to construct it is:

L = (sin(2 π (1 - m)) / (1 - m), (1 - cos(2 π (1 - m))) / (1 - m))

What is the curve when the circumference is opened? I observe closely how the sectors are opened (Figure 10 (a) to (c)) and conjecture that it should be an arc. At first it is a circle of centre (0, 1) and radius 1 (Figure 10(a)), and when it is laid down as a straight line it becomes an arc of radius infinitely long (Figure 10(c)).



(a) (b) (c)

Figure 10

Since the radius of the arc is 0 when $m = 0$ and is infinity when $m = 1$, I guess that the radius of this arc could be $\frac{1}{1-m}$. I look closely at the coordinates of $L = (\frac{\sin 2(1-m)\pi}{1-m}, \frac{1-\cos 2(1-m)\pi}{1-m})$ and suddenly realize that it is exactly the end point of the arc of radius $\frac{1}{1-m}$, centred at $(0, \frac{1}{1-m})$ and has the arc length 2π ! Therefore the opened circumference should be the arc of a circle centred at $(0, \frac{1}{1-m})$ with a radius $\frac{1}{1-m}$ whose two end points are (0, 0) and *L*. I therefore use the following command to construct the circumference *s*:

s = CircularArc[(0, 1 / (1 - m)), (0,0), L]

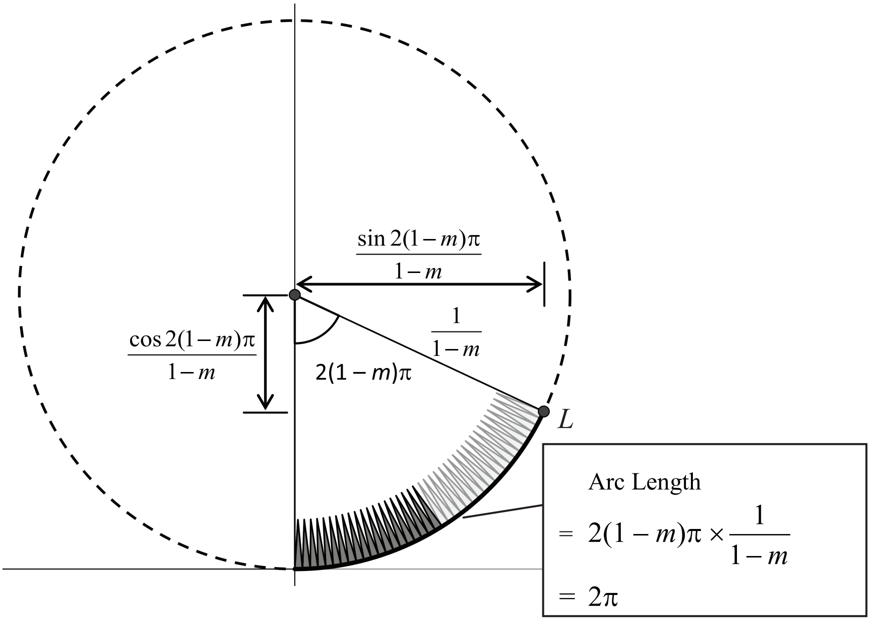


Figure 11

Finally, we note that when $m = 1$, s is undefined and the arc disappears. Since the circumference is straightened when $m = 1$, we can use the segment

Segment[(0 , 0), (2 pi , 0)]

to replace the disappeared arc, and here we end the discussion of the crucial parts of the design.

Mathematisation Using Information Technology

This article presents an attempt to implement Freudenthal's idea of mathematisation (Freudenthal, 1968, 1991) using information technology, in which a mathematical concept (the area of a circle) is re-invented in processes of exploration from intuitive mathematics (it is approximately equal to the area of the parallelogram) to sophisticated mathematics (it is exactly equal to the area of the limit rectangle) using tools that are more powerful than our predecessors possessed. We see how the use of the features and tools of GeoGebra can enable us to design tasks that empower students with enhanced abilities to acquire knowledge. It is hoped that through more carefully designed pedagogical tasks (with or without the use of technology), students can be more engaged in learning activities of mathematics, and mathematics can also be more engaging to them.

References

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